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Extensions of Suboptimal Output Feedback Control with Application to Large Space Structures

Daniel R. Hegg*
The Charles Stark Draper Laboratory, Inc., Cambridge, Mass.

Active control for vibration suppression may be required with future large flexible spacecraft. Order reduction of the structural model needed to allow active controller design can lead to rank deficiency in the reduced-state observation matrix. This violates an assumption of the Kosut method of suboptimal output feedback. Such rank deficiency is equivalent to redundancy of the sensor configuration relative to the reduced-order system. To avoid the need for ignoring certain measurements or altering the sensor configuration, the Kosut method is extended so as to be applicable with arbitrary sensor configurations. The main result is the discovery that the linear equation for the output feedback gain matrix is algebraically consistent regardless of the rank of the reduced-state observation matrix. When the latter is rank-deficient, a family of solutions to the gain equation exists. Free parameters generating this family are proportional in number to the rank-deficiency, and their values may be chosen to improve the performance of the full-order system driven by the reduced-order controller. A numerical example with a two-mass oscillator is given to demonstrate the application of the extensions and to indicate some of the possibilities for performance improvement.

Introduction

THE new class of space structures that are expected to emerge in the coming era of Shuttle-based space construction is characterized by significant increases in size and flexibility relative to current spacecraft. Structural vibration is certain to be a serious concern. For applications requiring precision attitude control doubts have been expressed as to the sufficiency of passive methods for vibration damping. This has motivated much recent research into appropriate methods for active control of structural vibrations.[1,2] For this class of structures, separation by frequency of the problems of rigid-body control and flexible-body control in an actual design is no longer possible.[2] However, the challenges of successful structural vibration damping are sufficient to merit separate study; such is the overall context of the present paper.

The results reported here were generated during a recent comparative study of several familiar output feedback approaches to linear multivariable control.[3] The purpose of that study was to evaluate the feasibility of these methods as candidates for designing reduced-order controllers for large space structures. Detailed questions of structural modeling and model order reduction are outside the scope of the present paper. It is assumed that a finite-element model of sufficiently high order is available for representing the structure, and that a selection of structural modes considered critical for the intended application has been made. Only the problem of reduced-order controller design is studied in detail.

A significant early contribution to linear multivariable control theory was the solution to the linear-quadratic regulator problem, in which the optimal closed-loop control is synthesized as a linear function of the state.[4] Recognizing that full-state measurements are not always possible, Levine and Athans[5] incorporated measurement constraints into the optimal regulator problem formulation. Although this approach is attractive, in that it avoids the need for state estimation in the feedback loop, convergence of the algorithm for solving the coupled nonlinear algebraic equations for the feedback gains cannot be guaranteed.[5,6] To sidestep this difficulty, Kosut[7] developed a suboptimal output feedback approach in which the feedback gains are obtainable without iteration. However, this result sacrifices the guarantee of stability for the closed-loop control system. (The connection between avoiding iteration and losing a guarantee of stability is a direct one.) Nevertheless, the Kosut approach has proved useful in determining an adequate guess for starting the Levine-Athans algorithm.[8]

The present paper reports the results of re-examining the Kosut suboptimal output feedback approach in the context of large space structure controller design. It was found that the assumption[7] of maximum rank for the observation matrix is inappropriate with reduced-order controllers, because the model-reduction process can easily lead to a violation of such a condition in the reduced-state model to be used for controller design. It is proved that the equation for the feedback gains[7] can be solved exactly, regardless of the rank of the observation matrix. In particular, when this matrix is rank-deficient (implying redundancy in the sensor configuration), a family of solutions for the gain matrix exists. An example with a two-mass oscillator having sensor redundancy demonstrates how free parameters defining the family of gain matrix solutions can be selected so as to improve the performance of the system relative to a corresponding (stable) system with fewer sensors.

To maintain a focus on the central results, discussion is restricted to only one of the Kosut approaches (minimum error excitation), and to the case of a centralized information structure. Essential features of the results are unaffected by considering the other Kosut approach (minimum norm) or a decentralized information structure.

Motivation

Finite-element representation is the only reasonable engineering tool capable of adequately modeling the dynamics of complex interconnected structures. However, the order of a model that is adequate for representing the structural dynamics must usually be reduced substantially before attempting to use the model as a plant representation for multivariable controller design. The model reduction process is a research topic in itself. The only point to be discussed here is that this process may increase the rank deficiency (i.e., the difference between maximum rank and actual rank) of the state-space observation matrix.

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*Technical Staff, Control and Dynamics Division. Member AIAA.

First, some notation is needed. A finite-element representation for an undamped vibrating structure can be expressed in modal coordinates as

$$\ddot{\eta}+\Omega^2\eta=\Phi^T B_A u \quad (1)$$

where $\eta\equiv(\eta_1,...,\eta_n)^T$ is the vector of modal coordinates; Ω^2 is the $n\times n$ diagonal matrix containing squares of the natural frequencies $\omega_1^2,...,\omega_n^2$; Φ is the $n\times n$ matrix whose columns are the structural mode shapes $\phi^1,...,\phi^n$; and B_A is the $n\times m$ actuator influence matrix for the control $u\equiv(u_1,...,u_m)^T$. Supercript T denotes matrix transpose. Similarly, the linear measurement equation has the form

$$y=C_P\Phi\eta+C_V\Phi\dot{\eta} \quad (2)$$

where $y\equiv(y_1,...,y_l)^T$ is the output vector, and C_P and C_V are the $l\times n$ displacement and rate measurement coefficient matrices, respectively. Without loss of generality, it can be assumed that the components of η have been reordered so that the first $N(N\leq n)$ of them are the selected critical modes $\eta_C\equiv(\eta_1,...,\eta_N)^T$, and the remainder are residual modes $\eta_R\equiv(\eta_{N+1},...,\eta_n)^T$. Corresponding submatrices Ω_C^2: $N\times N$, Ω_R^2: $(n-N)\times(n-N)$ of Ω^2 and Φ_C: $n\times N$, Φ_R: $n\times(n-N)$ of Φ are defined similarly. With $x\equiv(\eta_C,\dot{\eta}_C,\eta_R,\dot{\eta}_R)^T$ as a state vector, the structural model of Eqs. (1) and (2) becomes

$$\dot{x}=Ax+Bu \qquad y=Cx \quad (3)$$

where

$$A\equiv\left[\begin{array}{cc|cc} 0 & I_N & & 0 \\ -\Omega_C^2 & 0 & & \\ \hline & & 0 & I_{n-N} \\ & 0 & -\Omega_R^2 & 0 \end{array}\right] \qquad B\equiv\left[\begin{array}{c} 0 \\ \Phi_C^T B_A \\ \hline 0 \\ \Phi_R^T B_A \end{array}\right]$$

$$C\equiv[C_P\Phi_C \mid C_V\Phi_C \mid C_P\Phi_R \mid C_V\Phi_R]$$

The notation I_k denotes the $k\times k$ identity matrix. If residual mode truncation is used as the model reduction process, a reduced-order model with the state $x_C\equiv(\eta_C,\dot{\eta}_C)$ is obtained:

$$\dot{x}_C=A_Cx_C+B_Cu \qquad y_C=C_Cx_C \quad (4)$$

where

$$A_C\equiv\left[\begin{array}{c|c} 0 & I_N \\ \hline -\Omega_C^2 & 0 \end{array}\right] \quad B_C\equiv\left[\begin{array}{c} 0 \\ \hline \Phi_C^T B_A \end{array}\right] \quad C_C\equiv[C_P\Phi_C \mid C_V\Phi_C]$$

The reduced-order observation matrix C_C is obtained from the full-order observation matrix C by deleting the columns associated with the residual modes. However, there is no reason to expect the remaining set of columns to have the same rank as that of C, since the choice of critical modes is made on the basis of modal contributions to specified performance parameters rather than linear independence of structural mode shapes. In particular, C_C may be rank-deficient even if C is not; this is demonstrated by the example in the Application of Results section.

The Kosut method[7] assumes that the state-to-output observation matrix in the controller design model (in this case, the reduced-order observation matrix C_C) has maximum rank. Although it is always possible to satisfy this condition by sufficiently reducing the number of outputs to be considered, such a procedure may be undesirable in terms of system performance. This fact motivated the attempt to extend the applicability of the Kosut method to allow design model observation matrices having unrestricted deficiency in rank.

Main Results

The essential elements of the Kosut method (minimum error excitation) are briefly recalled, using the context of the reduced-order controller design model of Eqs. (4). An optimal pair (x_C^*,u^*) for system (4) is determined, using any criterion the designer chooses, except that it must lead to a relation of the form

$$u^*(t)=F^*x_C^*(t) \qquad (t\geq 0) \quad (5)$$

where F^* is such that $A_C+B_CF^*$ is asymptotically stable. The suboptimal problem relative to this optimal reference system is to find a pair (x_C,u) that satisfies Eqs. (4) together with $x_C(0)=x_C^*(0)$ and

$$u(t)=Gy_C(t) \qquad (t\geq 0)$$

(centralized output feedback) and that minimizes

$$J_W(G)\equiv\int_0^{+\infty}v^T(t)Wv(t)\mathrm{d}t \quad (6)$$

where $v(t)\equiv(GC_C-F^*)x_C^*(t)$ appears linearly in the forcing term of the differential equation for the error vector $e(t)\equiv x_C(t)-x_C^*(t)$, and W is a positive definite matrix chosen by the designer. Necessary conditions for suboptimality include a Liapunov matrix equation for a multiplier matrix P associated with the suboptimal cost of Eq. (6):

$$(A_C+B_CF^*)P+P(A_C+B_CF^*)^T+I_{2N}=0 \quad (7)$$

and the output feedback gain equation:

$$G(C_CPC_C^T)=F^*PC_C^T \quad (8)$$

Since the matrix $A_C+B_CF^*$ is asymptotically stable, the (unique) solution matrix P of Eq. (7) is positive definite.[9]

The following preliminary result connects the rank deficiency of the gain equation [more precisely, that of the coefficient matrix in Eq. (8)] with the rank deficiency of the reduced-order observation matrix. The latter is a direct indication of the redundancy of the sensor configuration.

Theorem 1. Assume that the matrix P:$2N\times 2N$ is positive definite. Then the product $C_CPC_C^T$ and the matrix C_C have the same rank.

Proof. Denote $r\equiv$rank (C_C). The result is immediate if $r=0$. Otherwise (Ref. 10, p. 103), there exist nonsingular matrices S:$l\times l$ and Q:$2N\times 2N$ such that

$$SC_CQ=\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]\equiv\hat{C}_C$$

Denoting $\hat{P}\equiv Q^{-1}P(Q^{-1})^T$, it follows that rank $(C_CPC_C^T)$ $=$rank $(\hat{C}_C\hat{P}\hat{C}_C^T)$. But the latter rank is r, since

$$\hat{C}_C\hat{P}\hat{C}_C^T=\left[\begin{array}{c|c} P_{11} & 0 \\ \hline 0 & 0 \end{array}\right]$$

where P_{11} is the $r\times r$ submatrix consisting of the first r rows and columns of the positive definite matrix $\hat{P}$ (Ref. 10, p. 170). ∎

The following theorem is the main result of the paper. It implies the algebraic consistency (i.e., exact solvability) of the gain equation (8) regardless of the rank of the reduced-order observation matrix.

Theorem 2. Assume that the matrix $P{:}2N\times 2N$ is positive definite. Then the augmented matrix

$$\begin{bmatrix} C_C P C_C^T \\ \hline F^* P C_C^T \end{bmatrix}$$

has the same rank as the coefficient matrix $C_C P C_C^T$.

Proof. Denote

$$E \equiv \begin{bmatrix} C_C \\ \hline F^* \end{bmatrix} P$$

Since the rank of a matrix is equal to the number of its linearly independent rows: rank $(C_C P C_C^T) \le$ rank (EC_C^T). Conversely, suppose that this inequality is strict. Denote $r \equiv$ rank (EC_C^T). Using Theorem 1, it follows that rank $(C_C) < r$. But then it follows easily (Ref. 10, p. 75) that

$$\text{rank}\ (EC_C^T) \le \min\{\text{rank}\ (E),\ \text{rank}\ (C_C^T)\} < r$$

which contradicts the definition of r. Hence, rank $(C_C P C_C^T) \ge$ rank (EC_C^T). ■

Algebraically consistent linear equations of the form $GA_0 = B_0$, such as Eq. (8), have infinitely many solutions when the coefficient matrix A_0 is rank-deficient. This may be expressed precisely by characterizing the structure of solutions to the associated homogeneous equation.

Theorem 3. Let the matrix $A_0{:}l\times 2N$ have rank r. Then a matrix $G^I{:}m\times l$ satisfies the homogeneous equation $GA_0 = 0$ if and only if it is a product of the form $G^I = \Gamma S$, where S: $(l-r)\times l$ is a full-rank matrix whose $l-r$ row vectors form a basis for the null space of A_0, and $\Gamma{:}m\times(l-r)$ is an arbitrary matrix.

Proof. Sufficiency is verifiable by direct calculation. To show necessity, only the case $m=1$ need be considered. Since A_0 has rank r, the null space of A_0 has rank $l-r$ (Ref. 10, p. 34). If $r=l$, the only solution is the zero vector. Otherwise, any solution is representable as a linear combination of basis vectors for the null space of A_0:

$$G = [\gamma_1 \cdots \gamma_{l-r}] \begin{bmatrix} s^1 \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline s^{l-r} \end{bmatrix} \qquad ■$$

Theorem 3 shows that the general solution to the gain equation (8) contains $m\cdot(l-r)$ arbitrary parameters, where r is the rank of $C_C P C_C^T$.

Theorems 1-3 constitute the extension to the Kosut design approach.[7] The main advance is the realization that the gain equation (8) is always solvable exactly. If the reduced-state observation matrix C_C has maximum rank, then Eq. (8) has the unique solution $G = (F^* P C_C^T)(C_C P C_C^T)^{-1}$ as reported by Kosut.[7] Otherwise, Eq. (8) has a family of solutions containing $m\cdot(l-r)$ arbitrary parameters, where $r=$ rank (C_C) (cf. Theorem 1), m is the number of actuator inputs to the reduced-order plant, and l is the number of sensor outputs from the plant. Judicious selections for values of these free parameters during the design process may improve the overall performance of the closed-loop system. The example in the next section illustrates the application of these results.

Application of Results

The extensions to the Kosut method developed above are applied to the design of an active controller for vibration suppression in the two-mass oscillator shown in Fig. 1. Physical constants for this system are: $m_1 = 1$ kg, $m_2 = 2$ kg, $k_1 = 1$ N/m, $k_2 = 4$ N/m. External dynamical variables are the displacements (from equilibrium) q_i, and the disturbance forces f_i, respectively, associated with mass m_i. Controller variables are the inputs u_i to actuator i and the outputs y_i from sensor i. Physical coordinates shown are related to the modal coordinates of Eq. (1) by the relation $q = \Phi\eta$.

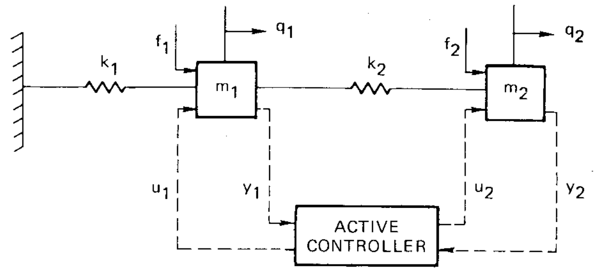

Fig. 1 Double spring-mass system.

This system has two vibration modes ($n=2$): symmetric (mode 1) and antisymmetric (mode 2). The parameters for the modal representation of Eq. (1) are: $\omega_1 = 0.5463$ s^{-1}, $\omega_2 = 2.589$ s^{-1}, $\phi^{1T} = (0.5155,\ 0.6059)$, $\phi^{2T} = (-0.8569,\ 0.3645)$. One force actuator is located on each mass; thus $m=2$ and $B_A = I_2$. The design objective is to incorporate 10% of critical damping into the antisymmetric mode via active feedback control; the symmetric mode is to be ignored (i.e., $N=1$). Hence, after reordering components: $\eta \equiv (\eta_C, \eta_R) = (\eta_2, \eta_1)$. Rate sensors only are to be employed; thus, $C_P = 0$ in the output equation (2). Initially, rate sensing of *each* mass element is considered.

Two-Sensor Configuration

With a rate sensor on each mass, $l=2$ and $C_V = I_2$ in Eq. (2). The full-state observation matrix in Eq. (3), $C \equiv [0\ \phi^2\ 0\ \phi^1]$, has rank 2, but the reduced-state observation matrix in Eq. (4), $C_C \equiv [0\ \phi^2]$, only has rank 1, showing how rank deficiency in C_C can occur in the model reduction process. The Kosut methods[7] cannot be used with this reduced-order model because of the rank deficiency; however, a design can be effected using the extensions developed in the present paper.

An optimal linear-quadratic regulator design[4] is used to determine a reference feedback structure [Eq. (5)] for the reduced-order model of Eqs. (4). Quadratic cost parameters are based on desired values for the parameters ζ and ω in the second-order closed-loop system: $Q = \text{diag}\ (q_{ii})$, $q_{11} = 1.044$, $q_{22} = 0.1596$, and $R = I_2$. The feedback matrix F^* is

$$F^* = \begin{bmatrix} 0.06639 & 0.5161 \\ -0.02824 & -0.2195 \end{bmatrix}$$

The second-order reference closed-loop system represented by $A_C + B_C F^*$ is stable, with $\zeta^* = 0.1004$, $\omega^* = 2.602$ s^{-1}.

For the suboptimal design, the solution of Eq. (7) is

$$P = \begin{bmatrix} 1.137 & -0.5 \\ -0.5 & 7.438 \end{bmatrix}$$

It can be easily shown that $F^* P C_C^T = -\sigma(C_C P C_C^T)$, where σ is an explicit function of ω_2, ϕ^2, ζ^*, and ω^*. Thus, Eq. (8) has a particular solution: $G = \text{diag}\ (-\sigma)$; using the reference system values for ζ^* and ω^* leads to $\sigma = \sigma^0 \equiv 0.5971$. The general solution of Eq. (8) can be expressed in terms of two arbitrary parameters:

$$G(\epsilon,\delta) = \begin{bmatrix} -\sigma + \epsilon\rho(\phi^2) & -\epsilon \\ -\delta & -\sigma + \delta/\rho(\phi^2) \end{bmatrix}$$

where $\rho(\phi^2) \equiv \phi_2^2/\phi_1^2$. The closed-loop reduced-order system of Eqs. (4) for the critical mode represented by $A_C + B_C[G(\epsilon,\delta)C_C]$ is *independent* of the free parameters, and is stable, with $\zeta_C = 0.1$, $\omega_C = \omega_2$. The full fourth-order system of Eqs. (3), incorporating the reduced-order controller, is represented by $A+B[G(\epsilon,\delta)C]$. The information is now available for selecting values of the free parameters ϵ and δ for desired system performance. Although the modal differential equations are coupled through the controller, selecting the free parameters to satisfy

$$\delta-\epsilon=\sigma(-\phi^{2T}\phi^{1})/\det\Phi \qquad (9)$$

removes the residual mode excitation from the differential equation for the critical mode. With this choice, the critical mode dynamics remain the same as with the reduced-order system. In particular, the effects of control spillover to the residual mode do not feed back into the critical mode dynamics. Use of Eq. (9) simplifies the stability analysis of the full system; stability is assured for

$$\sigma>0 \qquad \epsilon>-2(-\phi_l^2)\phi_2^2\sigma=-0.6247\sigma$$

The remaining degree of freedom in the choice of arbitrary parameters may be used to influence the damping factor for the residual mode. The choice

$$\epsilon^0=[\sqrt{2}-\sigma/\omega_l]2\omega_l(-\phi_l^2)\phi_2^2=0.1096$$

sets the coefficient of $\dot{\eta}_l$ in the residual mode equation to achieve $\zeta_l=1/\sqrt{2}$. With the corresponding value for δ^0 from Eq. (9), the design gain is

$$G^0\equiv G(\epsilon^0,\delta^0)=\begin{bmatrix}-0.6437 & -0.1096\\ -0.2961 & -1.293\end{bmatrix}$$

Figure 2 shows the time response in modal coordinates to a positive unit displacement of the outer mass. Figure 3 shows the frequency response of outer mass displacement due to a periodic disturbance on the same mass. Both responses are quite satisfactory.

One-Sensor Configuration

In the following, the number of rate sensor measurements to be considered in the design is reduced, so as to be able to employ the original Kosut approach.[7] It can be readily shown that the full system is unstable using the Kosut approach if the single rate sensor considered is the one on the outer mass. With a single rate sensor on the inner mass, $l=1$, $C_V=[1\ 0]$ in Eq. (2), and the full-state observation matrix in Eq. (3) is $C=[0\ \phi_l^2\ 0\ \phi_l^1]$. The reduced-state observation matrix in Eq. (4) is $C_C=[0\ \phi_l^2]$, which has maximum rank, allowing application of the Kosut approach.

The reference system may be taken as in the preceding design. Thus, F^* [Eq. (5)] and also P [Eq. (7)] remain the same. The gain equation (8) has the unique solution

$$G^0=\begin{bmatrix}-\sigma\\ -\sigma\rho(\phi^2)\end{bmatrix}$$

with $\sigma=\sigma^0$ as above. The closed-loop reduced-order system of Eqs. (4) for the critical mode represented by $A_C+B_C[G^0C_C]$ is stable, with $\zeta_C=0.1$, $\omega_C=\omega_2$ as before. The full fourth-order system of Eqs. (3), incorporating the reduced-order controller represented by $A+B[G^0C]$, is also stable, with $\zeta_2=0.1002$ (critical mode damping) and $\zeta_l=0.07272$ (residual mode damping). Figures 4 and 5 show the same characteristics of the one-sensor design as do Figs. 2 and 3, respectively, for the two-sensor design. Although the full system incorporating the one-sensor controller design is stable, its performance is not as good as that with the two-sensor controller, primarily due to the much lower damping in the residual mode.

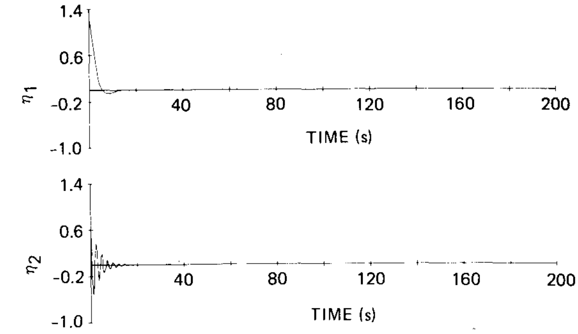

Fig. 2 Modal response to initial condition $q_2(0)=1$ (two-sensor design).

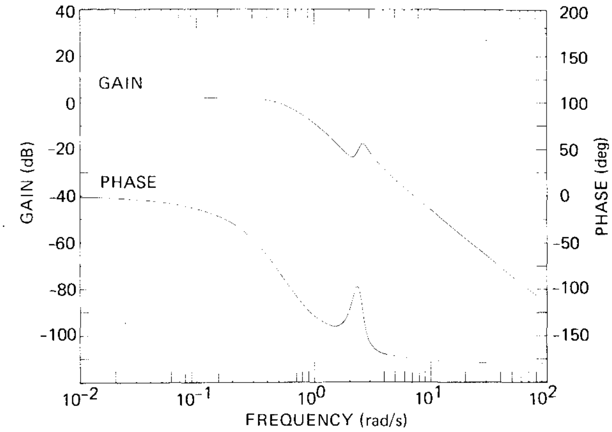

Fig. 3 Frequency response of q_2 to periodic disturbance f_2 (two-sensor design).

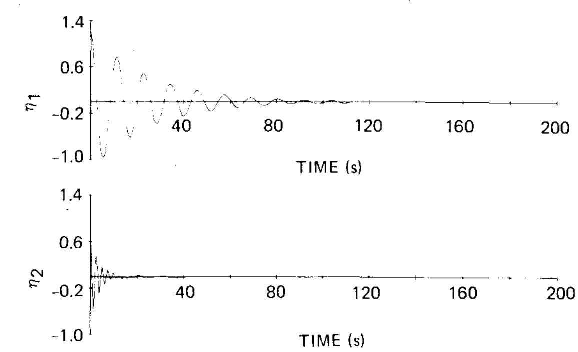

Fig. 4 Modal response to initial condition $q_2(0)=1$ (one-sensor design).

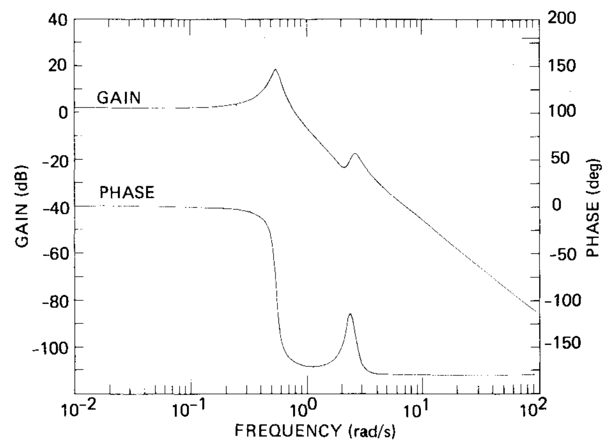

Fig. 5 Frequency response of q_2 to periodic disturbance f_2 (one-sensor design).

Conclusions

In this paper it has been noted that plant representations obtained by a process of model-order reduction may not satisfy the rank conditions required by the Kosut method of suboptimal output feedback control. Such conditions are attributable to redundancy in the sensor configuration relative to the reduced-order model. Extensions to the Kosut method have been developed, which can be applied with arbitrary sensor configurations. When a rank deficiency exists in the reduced-state observation matrix, a family of solutions for the suboptimal gain matrix exists. The number of free parameters defining this family is proportional to the rank deficiency; these parameters may often be selected so as to improve the performance of the full-order system driven by the reduced-order controller. Examples of performance improvement include partial decoupling (especially triangularization) of the modal differential equations, and the ability to influence the dynamics of certain residual modes significantly without explicitly including such modes in the design model.

The similarity of the Kosut method to the Levine-Athans method of optimal output feedback has been noted. In some cases, separate designs produced by the two methods are indistinguishable. Moreover, the extensions to the Kosut method reported in the present paper are equally applicable to each stage of the Levine-Athans algorithm. Whether this latter observation has any significance for the problem of convergence of that algorithm is an open question.

Acknowledgments

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